Multiple Polylogarithms: An Introduction

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Abstract

Multiple polylogarithms in a single variable are defined by

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

when $s_1,\ldots,s_k$ are positive integers and z a complex number in the unit disk. For $k=1$, this is the classical polylogarithm $\mathrm{Li}_s(z)$. These multiple polylogarithms can be defined also in terms of iterated Chen integrals and satisfy *shuffle relations*. Multiple polylogarithms in several variables are defined for $s_i \geq 1$ and $|z_i| < 1$ $(1 \leq i \leq k)$ by

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}},$$

and they satisfy not only shuffle relations, but also *stuffle relations*. When one specializes the shuffle relations in one variable at $z=1$ and the stuffle relations in several variables at $z_1=\cdots=z_k=1$, one gets linear or quadratic dependence relations between the Multiple Zeta Values

$$\zeta(s_1,\ldots,s_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

which are defined for k, $s_1,\ldots,s_k$ positive integers with $s_1 \geq 2$. The *Main Diophantine Conjecture* states that one obtains in this way all algebraic relations between these MZV.

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0. Introduction

A long term project is to determine all algebraic relations among the values

$$\pi,\ \zeta(3),\ \zeta(5), \dots, \zeta(2n+1), \dots$$

of the Riemann zeta function

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}\cdot$$

So far, one only knows that the first number in this list, π, is transcendental, that the second one, $\zeta(3)$, is irrational, and that the other ones span a $\mathbb{Q}$-vector space of infinite dimension [R], [BR].

The expected answer is disappointingly simple: it is widely believed that there are no relations, which means that these numbers should be algebraically independent:

(?) *For any $n \geq 0$ and any nonzero polynomial $P \in \mathbb{Z}[X_0, \dots, X_n]$,*

$$P\bigl(\pi, \zeta(3), \zeta(5), \dots, \zeta(2n+1)\bigr) \neq 0.$$

If true, this property would mean that there is no interesting algebraic structure.

The situation changes drastically if we enlarge our set so as to include the so-called Multiple Zeta Values (MZV, also called Euler-Zagier numbers – see [Eu] and [Z]):

$$\zeta(s_1, \dots, s_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} \cdots n_k^{s_k}},$$

which are defined for k, $s_1, \dots, s_k$ positive integers with $s_1 \geq 2$. It may be hoped that the initial goal could be reached if one could determine all algebraic relations between the MZV. Now there are plenty of relations between them, providing a rich algebraic structure. One type of such relations arises when one multiplies two such series: it is easy to see that one gets a linear combination of MZV. There is another type of algebraic relations between MZV, coming from their expressions as integrals. Again the product of two such integrals is a linear combination of MZV. Following [B^3], we will use the name *stuffle* for the relations arising from the series, and *shuffle* for those arising from the integrals.

The *Main Diophantine Conjecture* (Conjecture 5.3 below) states that these relations are sufficient to describe all algebraic relations between MZV. One should be careful when stating such a conjecture: it is necessary to include some relations which are deduced from the stuffle and shuffle applied to divergent series (i.e. with $s_1 = 1$).

There are several ways of dealing with the divergent case. Here, we use the *multiple polylogarithms*

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}},$$

which are defined for $|z|<1$ when $s_1,\ldots,s_k$ are all ≥ 1, and which are also defined for $|z|=1$ if $s_1\geq 2$.

These multiple polylogarithms can be expressed as iterated Chen integrals, and from this representation one deduces shuffle relations. There is no stuffle relations for multiple polylogarithms in a single variable, but one recovers them by introducing the multivariables functions

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}} \qquad (s_i\geq 1,\ 1\leq i\leq k)$$

which are defined not only for $|z_i|<1$ $(1\leq i\leq k)$, but also for $|z_i|\leq 1$ $(1\leq i\leq k)$ if $s_1\geq 2$.

Notation. Given a string $a_1,\ldots,a_k$ of integers, the notation $\{a_1,\ldots,a_k\}_n$ stands for the kn-tuple

$$(a_1,\ldots,a_k,\ldots,a_1,\ldots,a_k),$$

where the string $a_1,\ldots,a_k$ is repeated n times.

1. Multiple Polylogarithms in One Variable and Multiple Zeta Values

Let k, $s_1,\ldots,s_k$ be positive integers. Write $\underline{s}$ in place of $(s_1,\ldots,s_k)$. One defines a complex function of one variable by

$$\mathrm{Li}_{\underline{s}}(z) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z^{n_1}}{n_1^{s_1}\cdots n_k^{s_k}}.$$

This function is analytic in the open unit disk, and, in the case $s_1\geq 2$, it is also continuous on the closed unit disk. In the latter case we have

$$\zeta(\underline{s}) = \mathrm{Li}_{\underline{s}}(1).$$

One can also define in an equivalent way these functions by induction on the number $p=s_1+\cdots+s_k$ (the *weight* of $\underline{s}$) as follows. Plainly we have

$$(1.1)\qquad z\frac{d}{dz}\mathrm{Li}_{(s_1,\ldots,s_k)}(z) = \mathrm{Li}_{(s_1-1,s_2,\ldots,s_k)}(z) \quad \text{if} \quad s_1\geq 2$$

and

$$(1.2) \qquad (1-z)\frac{d}{dz}\mathrm{Li}_{(1,s_2,\ldots,s_k)}(z) = \mathrm{Li}_{(s_2,\ldots,s_k)}(z).$$

Together with the initial conditions

$$(1.3) \qquad \mathrm{Li}_{\underline{s}}(0) = 0,$$

the differential equations (1.1) and (1.2) determine all the $\mathrm{Li}_{\underline{s}}$.

Therefore, as observed by M. Kontsevich (cf. [Z]; see also [K] Chap. XIX, § 11 for an early reference to H. Poincaré, 1884), an equivalent definition for $\mathrm{Li}_{\underline{s}}$ is given by integral formulae as follows. Starting(*) with $k = s = 1$, we write

$$\mathrm{Li}_1(z) = -\log(1-z) = \int_0^z \frac{dt}{1-t},$$

where the complex integral is over any path from 0 to z inside the unit circle. From the differential equations (1.1) one deduces, by induction, for $s \geq 2$,

$$\mathrm{Li}_s(z) = \int_0^z \mathrm{Li}_{s-1}(t)\frac{dt}{t} = \int_0^z \frac{dt_1}{t_1}\int_0^{t_1}\frac{dt_2}{t_2}\cdots\int_0^{t_{s-2}}\frac{dt_{s-1}}{t_{s-1}}\int_0^{t_{s-1}}\frac{dt_s}{1-t_s}\cdot$$

In the last formula, the complex integral which is written on the left (and which is the last to be computed) is over any path inside the unit circle from 0 to z, the second one is from 0 to t_1,... and the last one on the right, which is the first to be computed, is from 0 to t_{s-1}.

Chen iterated integrals (see [K] Chap. XIX, § 11) provide a compact form for such expressions as follows. For $\varphi_1, \ldots, \varphi_p$ differential forms and x, y complex numbers, define inductively

$$\int_x^y \varphi_1\cdots\varphi_p = \int_x^y \varphi_1(t)\int_x^t \varphi_2\cdots\varphi_p.$$

For $\underline{s} = (s_1, \ldots, s_k)$, set

$$\omega_{\underline{s}} = \omega_0^{s_1-1}\omega_1\cdots\omega_0^{s_k-1}\omega_1,$$

where

$$\omega_0(t) = \frac{dt}{t} \quad \text{and} \quad \omega_1(t) = \frac{dt}{1-t}\cdot$$

(*) This induction could as well be started from $k = 0$, provided that we set $\mathrm{Li}_{\emptyset}(z) = 1$.

Then the differential equations (1.1) and (1.2) with initial conditions (1.3) can be written

$$\mathrm{Li}_{\underline{s}}(z) = \int_0^z \omega_{\underline{s}}. \tag{1.4}$$

`Example.` For any $n \geq 1$ and $|z| < 1$ we have

$$\mathrm{Li}_{\{1\}_n}(z) = \frac{1}{n!}\Big(\log\big(1/(1-z)\big)\Big)^n, \tag{1.5}$$

which can be written in terms of generating series as

$$\sum_{n=0}^{\infty} \mathrm{Li}_{\{1\}_n}(z) x^n = (1-z)^{-x}.$$

The constant term $\mathrm{Li}_{\{1\}_0}(z)$ is 1.

2. Shuffle Product and the First Standard Relations

Denote by $X = \{\omega_0, \omega_1\}$ the alphabet with two letters and by X^* the set of words on X. A word is nothing else than a non-commutative monomial in the two letters ω_0 and ω_1. The linear combinations of such words with rational coefficients

$$\sum_u c_u u,$$

where $\{c_u \; ; \; u \in X^*\}$ is a set of rational numbers with finite support, is the non-commutative ring $\mathbb{Q}\langle\omega_0, \omega_1\rangle$. We are interested with the set $X^*\omega_1$ of words which end with ω_1, together with the empty word $\emptyset$. The linear combinations of such words is a left ideal of $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ which we denote by $\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$.

The set $X^*\omega_1$ is also the set of words $\omega_{\underline{s}}$, with $\underline{s} = (s_1, \ldots, s_k)$. We define $\mathrm{Li}_u(z)$ for $u \in X^*\omega_1$ by $\mathrm{Li}_u(z) = \mathrm{Li}_{\underline{s}}(z)$ when $u = \omega_{\underline{s}}$. By linearity we extend the definition of $\mathrm{Li}_u(z)$ to the ideal $\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$:

$$\mathrm{Li}_v(z) = \sum_u c_u \mathrm{Li}_u(z) \quad \text{for} \quad v = \sum_u c_u u.$$

The set of *convergent words* is the set, denoted by $\omega_0 X^* \omega_1$, of words which start with ω_0 and end with ω_1 (including the empty word). The $\mathbb{Q}$-vector subspace they span in $\mathbb{Q}\langle\omega_0, \omega_1\rangle$ is denoted by $\omega_0\mathbb{Q}\langle\omega_0, \omega_1\rangle\omega_1$, and for v in this space we set

$$\zeta(v) = \mathrm{Li}_v(1).$$

`Definition.` The shuffle product of two words in X^* is the element in $\mathbb{Q}\langle\omega_0,\omega_1\rangle$ which is defined inductively as follows:

$$\emptyset ⧢ u = u ⧢ \emptyset = u$$

for any u in $X^*\omega_1$, and

$$(\omega_i u) ⧢ (\omega_j v) = \omega_i(u ⧢ \omega_j v) + \omega_j(\omega_i u ⧢ v)$$

for u, v in X^* and i, j equal to 0 or 1.

This product is extended bilinearly to $\mathbb{Q}\langle\omega_0,\omega_1\rangle$ and defines a commutative and associative law. Moreover $\mathbb{Q}\langle\omega_0,\omega_1\rangle\omega_1$ is stable under ⧢.

Computing the product $\mathrm{Li}_u(z)\mathrm{Li}_{u'}(z)$ of the two associated Chen iterated integrals yields (see [MPH], Th. 2):

Proposition 2.1. *For u and u' in $X^*\omega_1$,*

$$\mathrm{Li}_u(z)\mathrm{Li}_{u'}(z) = \mathrm{Li}_{u ⧢ u'}(z).$$

For instance from

$$\omega_1 ⧢ \omega_0\omega_1 = \omega_1\omega_0\omega_1 + 2\omega_0\omega_1^2$$

we deduce

$$\mathrm{Li}_1(z)\mathrm{Li}_2(z) = \mathrm{Li}_{1,2}(z) + 2\mathrm{Li}_{2,1}(z). \tag{2.2}$$

Setting $z = 1$, we deduce from Proposition 2.1:

$$\zeta(u)\zeta(u') = \zeta(u ⧢ u') \tag{2.3}$$

for u and u' in $\omega_0 X^*\omega_1$.

These are the *first standard relations* between multiple zeta values.

3. Shuffle Product for Multiple Polylogarithms in Several Variables

The functions of k complex variables(*)

$$\mathrm{Li}_{\underline{s}}(z_1,\ldots,z_k) = \sum_{n_1>n_2>\cdots>n_k\geq 1} \frac{z_1^{n_1}\cdots z_k^{n_k}}{n_1^{s_1}\cdots n_k^{s_k}}$$

(*) Our notation for

$$\mathrm{Li}_{(s_1,\ldots,s_k)}(z_1,\ldots,z_k)$$

is the same as in [W], but it corresponds to Goncharov's notation [G2] for

$$\mathrm{Li}_{(s_k,\ldots,s_1)}(z_k,\ldots,z_1).$$

have been considered as early as 1904 by N. Nielsen, and rediscovered later by A.B. Goncharov [G1,G2]. Recently, J. Écalle [É] used them for z_i roots of unity (in case $s_1 \geq 2$): these are the *decorated multiple polylogarithms*. Of course one recovers the one variable functions $\mathrm{Li}_{\underline{s}}(z)$ by specializing $z_2 = \cdots = z_k = 1$. For simplicity we write $\mathrm{Li}_{\underline{s}}(\underline{z})$, where $\underline{z}$ stands for $(z_1, \ldots, z_k)$. There is an integral formula which extends (1.4). Define

$$\omega_z(t) = \begin{cases} \dfrac{z dt}{1 - zt} & \text{if } z \neq 0, \\ \dfrac{dt}{t} & \text{if } z = 0. \end{cases}$$

From the differential equations

$$z_1 \frac{\partial}{\partial z_1} \mathrm{Li}_{\underline{s}}(\underline{z}) = \mathrm{Li}_{(s_1 - 1, s_2, \ldots, s_k)}(\underline{z}) \quad \text{if} \quad s_1 \geq 2$$

and

$$(1 - z_1) \frac{\partial}{\partial z_1} \mathrm{Li}_{(1, s_2, \ldots, s_k)}(\underline{z}) = \mathrm{Li}_{(s_2, \ldots, s_k)}(z_1 z_2, z_3, \ldots, z_k),$$

generalizing (1.1) and (1.2), we deduce

$$\mathrm{Li}_{\underline{s}}(\underline{z}) = \int_0^1 \omega_0^{s_1 - 1} \omega_{z_1} \omega_0^{s_2 - 1} \omega_{z_1 z_2} \cdots \omega_0^{s_k - 1} \omega_{z_1 \cdots z_k}.$$

Because of the occurrence of the products $z_1 \cdots z_j$ $(1 \leq j \leq k)$, the authors of [G1] and [B^3L] perform the change of variables

$$y_j = z_1^{-1} \cdots z_j^{-1} \quad (1 \leq j \leq k) \quad \text{and} \quad z_j = \frac{y_{j-1}}{y_j} \quad (1 \leq j \leq k)$$

with $y_0 = 1$, and introduce the differential forms

$$\omega'_y(t) = -\omega_{y^{-1}}(t) = \frac{dt}{t - y},$$

so that $\omega'_0 = \omega_0$ and $\omega'_1 = -\omega_1$. Also they define

$$\begin{aligned} \lambda \binom{s_1, \ldots, s_k}{y_1, \ldots, y_k} &= \mathrm{Li}_{\underline{s}}(1/y_1, y_1/y_2, \ldots, y_{k-1}/y_k) \\ &= \sum_{\nu_1 \geq 1} \cdots \sum_{\nu_k \geq 1} \prod_{j=1}^{k} y_j^{-\nu_j} \left(\sum_{i=j}^{k} \nu_i \right)^{-s_j} \\ &= (-1)^p \int_{\Delta_p} \omega_0^{s_1 - 1} \omega'_{y_1} \cdots \omega_0^{s_k - 1} \omega'_{y_k}. \end{aligned}$$

With this notation some formulae are simpler. For instance the shuffle relation is easier to write with λ: the shuffle is defined on words on the alphabet $\{\omega'_y \ ; \ y \in \mathbb{C}\}$, (including $y = 0$), inductively by

$$(\omega'_y u) ⧢ (\omega'_{y'} v) = \omega'_y (u ⧢ \omega'_{y'} v) + \omega'_{y'} (\omega'_y u ⧢ v).$$

4. Stuffle Product and the Second Standard Relations

The functions $\mathrm{Li}_{\underline{s}}(\underline{z})$ satisfy not only shuffle relations, but also *stuffle relations* arising from the product of two series:

$$\mathrm{Li}_{\underline{s}}(\underline{z})\mathrm{Li}_{\underline{s}'}(\underline{z}') = \sum_{\underline{s}''} \mathrm{Li}_{\underline{s}''}(\underline{z}''), \tag{4.1}$$

where the notation is as follows: $\underline{s}''$ runs over the tuples $(s''_1, \dots, s''_{k''})$ obtained from $\underline{s} = (s_1, \dots, s_k)$ and $\underline{s}' = (s'_1, \dots, s'_{k'})$ by inserting, in all possible ways, some 0 in the string $(s_1, \dots, s_k)$ as well as in the string $(s'_1, \dots, s'_{k'})$ (including in front and at the end), so that the new strings have the same length k'', with $\max\{k, k'\} \le k'' \le k + k'$, and by adding the two sequences term by term. For each such $\underline{s}''$, the component z''_i of $\underline{z}$ is z_j if the corresponding s''_i is just s_j (corresponding to a 0 in $\underline{s}'$), it is z'_ℓ if the corresponding s''_i is s'_ℓ (corresponding to a 0 in $\underline{s}$), and finally it is $z_j z'_\ell$ if the corresponding s''_i is $s_j + s'_\ell$. For instance

$$\begin{array}{cccccccc}
\underline{s} & s_1 & s_2 & 0 & s_3 & s_4 & \cdots & 0 \\
\underline{s}' & 0 & s'_1 & s'_2 & 0 & s'_3 & \cdots & s'_{k'} \\
\underline{s}'' & s_1 & s_2 + s'_1 & s'_2 & s_3 & s_4 + s'_3 & \cdots & s'_{k'} \\
\underline{z}'' & z_1 & z_2 z'_1 & z'_2 & z_3 & z_4 z'_3 & \cdots & z'_{k'}.
\end{array}$$

Of course the 0's are inserted so that no s''_i is zero.

`Examples.` For $k = k' = 1$ the stuffle relation (4.1) yields

$$\mathrm{Li}_s(z)\mathrm{Li}_{s'}(z') = \mathrm{Li}_{(s,s')}(z, z') + \mathrm{Li}_{(s',s)}(z', z) + \mathrm{Li}_{s+s'}(zz'), \tag{4.2}$$

while for $k = 1$ and $k' = 2$ we have

$$\begin{aligned} \mathrm{Li}_s(z)\mathrm{Li}_{(s'_1,s'_2)}(z'_1, z'_2) &= \mathrm{Li}_{(s,s'_1,s'_2)}(z, z'_1, z'_2) + \mathrm{Li}_{(s'_1,s,s'_2)}(z'_1, z, z'_2) \\ &+ \mathrm{Li}_{(s'_1,s'_2,s)}(z'_1, z'_2, z) + \mathrm{Li}_{(s+s'_1,s'_2)}(zz'_1, z'_2) + \mathrm{Li}_{(s'_1,s+s'_2)}(z'_1, zz'_2). \end{aligned} \tag{4.3}$$

The *stuffle product* is defined on $X^*\omega_1$ inductively by

$$\emptyset * u = u * \emptyset = u$$

for $u \in X^*\omega_1$ and

$$\begin{aligned} &(\omega_0^{s-1}\omega_1 u) * (\omega_0^{t-1}\omega_1 u') = \\ &\qquad \omega_0^{s-1}\omega_1(u * \omega_0^{t-1}\omega_1 u') + \omega_0^{t-1}\omega_1(\omega_0^{s-1}\omega_1 u * u') + \omega_0^{s+t-1}\omega_1(u * u') \end{aligned}$$

for u and u' in $X^*\omega_1$, $s \ge 1$, $t \ge 1$.

Specializing (4.1) at $z_1 = \cdots = z_k = z'_1 = \cdots = z'_{k'} = 1$, we deduce

$$\zeta(u)\zeta(u') = \zeta(u * u') \tag{4.4}$$

for u and u' in $\omega_0 X^* \omega_1$.

These are the *second standard relations* between multiple zeta values. For instance (4.3) with $z = z'_1 = z'_2 = 1$ gives

$$\begin{aligned}\zeta(s)\zeta(s'_1, s'_2) = \zeta(s, s'_1, s'_2) + \zeta(s'_1, s, s'_2) + \zeta(s'_1, s'_2, s)\\ +\zeta(s + s'_1, s'_2) + \zeta(s'_1, s + s'_2)\end{aligned}$$

for $s \geq 2$, $s'_1 \geq 2$ and $s'_2 \geq 1$.

5. The Third Standard Relations and the Main Diophantine Conjectures

We start with an example. Combining the stuffle relation (4.2) for $s = s' = 1$ with the shuffle relation (2.2) for $z' = z$, we deduce

$$\mathrm{Li}_{1,2}(z, 1) + 2\mathrm{Li}_{2,1}(z, 1) = \mathrm{Li}_{1,2}(z, z) + \mathrm{Li}_{2,1}(z, z) + \mathrm{Li}_3(z^2). \tag{5.1}$$

The two sides are analytic inside the unit circle, but not convergent at $z = 1$. We claim that

$$F(z) = \mathrm{Li}_{1,2}(z, 1) - \mathrm{Li}_{1,2}(z, z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}(1 - z^{n_2})}{n_1 n_2^2}$$

tends to 0 as z tends to 1 inside the unit circle. Indeed for $|z| < 1$ we have

$$|1 - z^{n_2}| = |(1 - z)(1 + z + \cdots + z^{n_2 - 1})| < n_2 |1 - z|,$$

hence

$$\sum_{n_2=1}^{n_1-1} \frac{|1 - z^{n_2}|}{n_2^2} < |1 - z| \sum_{n_2=1}^{n_1-1} \frac{1}{n_2}.$$

From (1.5) with $n = 2$ we deduce

$$|F(z)| \leq |1 - z| \mathrm{Li}_{1,1}(|z|) = \frac{1}{2}|1 - z| \Big(\log\big(1/(1 - |z|)\big)\Big)^2.$$

Therefore, taking the limit of the relation (5.1) as $z \to 1$ yields Euler's formula

$$\zeta(2, 1) = \zeta(3).$$

This argument works in a quite general setting and yields the relations

$$\zeta(\omega_1 * u - \omega_1 ⧢ u) = 0 \tag{5.2}$$

for each $u \in \omega_0 X^* \omega_1$.

These are the *third standard relations* between multiple zeta values.

Zagier, Goncharov, Kontsevich,... (see [Z]) conjecture that the three standard relations (2.3), (4.4) and (5.2) constitute a basis of the ideal of algebraic relations between all numbers $\zeta(\underline{s})$. Here are precise statements.

We introduce independent variables Z_u, where u ranges over the set $X^*\omega_1$. For $v = \sum_u c_u u$ in $\mathbb{Q}\langle\omega_0,\omega_1\rangle\omega_1$, we set

$$Z_v = \sum_u c_u Z_u.$$

In particular for u_1 and u_2 in $\omega_0 X^* \omega_1$, $Z_{u_1 ⧢ u_2}$ and $Z_{u_1 * u_2}$ are linear forms in Z_u, $u \in \omega_0 X^* \omega_1$. Also, for $v \in \omega_0 \mathbb{Q}\langle\omega_0,\omega_1\rangle\omega_1$, $Z_{\omega_1 ⧢ v - \omega_1 * v}$ is a linear form in Z_u, $u \in \omega_0 X^* \omega_1$.

Denote by R the ring of polynomials with coefficients in $\mathbb{Q}$ in the variables Z_u where u ranges over the set $\omega_0 X^* \omega_1$, and by $\mathfrak{I}$ the ideal of R consisting of all polynomials which vanish under the specialization map

$$Z_u \mapsto \zeta(u) \quad (u \in \omega_0 X^* \omega_1).$$

Conjecture 5.3. *The polynomials*

$$Z_u Z_v - Z_{u ⧢ v}, \qquad Z_u Z_v - Z_{u*v} \quad \text{and} \quad Z_{\omega_1 ⧢ u - \omega_1 * u},$$

where u and v range over the set of elements in $\omega_0 X^* \omega_1$, generate the ideal $\mathfrak{I}$.

Denote by $\mathfrak{Z}_p$ the $\mathbb{Q}$-vector subspace of $\mathbb{R}$ spanned by the real numbers $\zeta(\underline{s})$ with $\underline{s}$ of weight p, with $\mathfrak{Z}_0 = \mathbb{Q}$ and $\mathfrak{Z}_1 = \{0\}$. Using any of the first two standard relations (2.3) or (4.4), one deduces $\mathfrak{Z}_p \cdot \mathfrak{Z}_{p'} \subset \mathfrak{Z}_{p+p'}$. This means that the $\mathbb{Q}$-vector subspace $\mathfrak{Z}$ of $\mathbb{R}$ spanned by all $\mathfrak{Z}_p$, $p \geq 0$, is a subalgebra of $\mathbb{R}$ over $\mathbb{Q}$ which is graded by the weight. From Conjecture 5.3 one deduces the following conjecture of Goncharov [G1]:

Conjecture 5.4. *As a $\mathbb{Q}$-algebra, $\mathfrak{Z}$ is the direct sum of $\mathfrak{Z}_p$ for $p \geq 0$.*

The dimension d_p of $\mathfrak{Z}_p$ satisfies $d_0 = 1$, $d_1 = 0$, $d_2 = d_3 = 1$. The expected value for d_p is given by a conjecture of Zagier [Z]:

Conjecture 5.5. *For $p \geq 3$ we have*

$$d_p = d_{p-2} + d_{p-3}.$$

An interesting question is whether Conjecture 5.3 implies Conjecture 5.5. For this question as well as other related problems, see [É].

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